

# THE SEMISIMPLE CONJUGACY CLASSES IN THE SYMPLECTIC GROUPS

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ABSTRACT. We determine the conjugacy classes of semisimple elements in the symplectic groups  $\mathrm{Sp}_{2m}(F)$ , where  $F$  is an arbitrary field of characteristic not 2. This note was originally a letter dated 23 March, 2006, from G.E. Wall to Cheryl Praeger, and has been reproduced with his kind permission.

## 1. THE GENERAL PROBLEM

The problem in question is to determine the conjugacy classes in the symplectic groups  $\mathrm{Sp}_{2m}(F)$  over a field  $F$ . The general method proposed in the present section is used in Section 2 to give a detailed (and elementary) account of the conjugacy classes of semisimple elements in the case where  $\mathrm{char} F \neq 2$ . (A *semisimple* element is one whose minimal polynomial is separable. These include all elements of finite order when  $\mathrm{char} F = 0$ .)

Denote by  $\mathcal{F}$  the set of all non-degenerate alternating bilinear forms

$$f: F^{2m} \times F^{2m} \rightarrow F$$

and by  $\mathcal{G}$  the general linear group  $\mathrm{GL}_{2m}(F)$  of all nonsingular linear mappings

$$T: F^{2m} \rightarrow F^{2m}.$$

The natural permutation action of  $\mathcal{G}$  on  $\mathcal{F}$  is defined by

$$(fT)(u, v) = f(uT, vT) \text{ for all } u, v \in F^{2m}.$$

The subgroup of  $\mathcal{G}$  formed by those elements that fix a given  $f$  is the symplectic group  $\mathrm{Sp}(f)$ . Since  $\mathcal{G}$  acts transitively on  $\mathcal{F}$ , these symplectic groups form a complete set of conjugate subgroups of  $\mathcal{G}$  (thereby justifying the generic notation  $\mathrm{Sp}_{2m}(F)$ ).

In order to put forms and linear mappings on the same footing, we introduce the set of pairs

$$\mathcal{P} = \{(f, T) \mid f \in \mathcal{F}, T \in \mathrm{Sp}(f)\}$$

and define the action of  $\mathcal{G}$  on  $\mathcal{P}$  by

$$(f, T)S = (fS, S^{-1}TS).$$

The crucial observation is this:

**Observation 1.1.** For fixed  $f_0 \in \mathcal{F}$ , the elements  $T_1, T_2, \dots \in \mathcal{G}$  are a set of representatives for the conjugacy classes of  $\mathrm{Sp}(f_0)$  if and only if  $(f_0, T_1), (f_0, T_2), \dots \in \mathcal{P}$  are a set of representatives for the orbits under the action of  $\mathcal{G}$  on  $\mathcal{P}$ .

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In short, the original conjugacy class problem can be reformulated as one about orbits on  $\mathcal{P}$ . It is from this new viewpoint that the problem will be treated from here. We now describe an alternative way of constructing a set of orbit representatives.

**Step 1:** We first choose representative elements  $R_1, R_2, \dots$  from the conjugacy classes  $\mathcal{K}_1, \mathcal{K}_2, \dots$  of  $\mathcal{G}$ , determining at the same time their centralisers  $b_1, b_2, \dots$  in  $\mathcal{G}$ . This is a matter of standard linear algebra.

**Step 2:** We next determine, for each such representative  $R_k$ , the set

$$\mathcal{F}_k = \{f \in \mathcal{F} \mid fR_k = f\} = \{f \in \mathcal{F} \mid (f, R_k) \in \mathcal{P}\}.$$

It may happen that  $\mathcal{F}_k$  is empty, which simply means that no symplectic group contains elements of  $\mathcal{G}$  conjugate to  $R_k$ . Assume now that  $\mathcal{F}_k$  is nonempty.

**Step 3:** The centraliser  $b_k$  acts naturally as a permutation group on  $\mathcal{F}_k$ . The final step is to determine a set of representatives  $f_{k1}, f_{k2}, \dots$  for the orbits of  $b_k$  in this action.

The pairs

$$(f_{11}, R_1), (f_{12}, R_1), \dots, (f_{21}, R_2), (f_{22}, R_2), \dots$$

so constructed form an alternative set of representatives for the orbits under the action of  $\mathcal{G}$  on  $\mathcal{P}$ , and thus give a new way of determining the conjugacy classes in the symplectic group.

## 2. SEMISIMPLE ELEMENTS

In order to avoid exceptional cases, we assume throughout that

$$(1) \quad \text{char } F \neq 2.$$

No further restriction is imposed for the present.

The first task (Step 2 of §1) is as follows: given a nonsingular, even-dimensional linear transformation over the field  $F$ , it is required to determine the nonsingular alternating bilinear forms that it leaves invariant.

In matrix terms, we are given  $X \in \text{GL}_{2m}(F)$  and are required to determine those  $A \in \text{GL}_{2m}(F)$  such that

$$(2) \quad A = -A', \quad A = XAX',$$

where  $'$  denotes transpose. Notice that these conditions are equivalent to

- (i) the form  $f_A$  given by  $f_A(u, v) = uAv'$  lies in  $\mathcal{F}$ , and
- (ii)  $X$  leaves  $f_A$  invariant, so that  $(f_A, X) \in \mathcal{P}$ .

The second task (Step 3 of §1) arises when the set of  $A$  in (2) is nonempty. The centraliser of  $X$  in  $\text{GL}_{2m}(F)$  acts on this set by congruence:

$$(3) \quad A \mapsto YAY' \text{ for } Y \text{ such that } Y^{-1}XY = X,$$

and it is required to determine a set of representatives

$$(4) \quad A_1, A_2, \dots$$

for the orbits. It is tacitly assumed from now on that  $A$  and  $X$  are nonsingular matrices satisfying (2).

**Lemma 2.1.**  *$X$  is similar to  $X^{-1}$ .*

*Proof.* By (2),  $A^{-1}X^{-1}A = X'$ , so that  $X^{-1}$  is similar to  $X'$  and hence to  $X$ .  $\square$

**Notation 2.2.** If  $f(t)$  is a monic polynomial with  $f(0) \neq 0$  then  $f^-(t)$  denotes the monic polynomial whose roots are the reciprocals of those of  $f(t)$ . Let  $c_Y(t)$  denote the characteristic polynomial of a square matrix  $Y$ .

**Definition 2.3.** An *elementary divisor* of a square matrix  $Y$  is a divisor of the minimal polynomial of  $Y$  of the form  $f(t) = g(t)^\lambda$ , where  $g(t)$  is monic and irreducible, which is related to the rational canonical form of  $Y$ .

Later we shall assume that  $Y$  is semisimple. In this case, irreducible factors of the minimal polynomial of  $Y$  occur with multiplicity 1.

**Corollary 2.4.** *If  $f(t)$  is an elementary divisor of  $X$ , then  $f^-(t)$  is an elementary divisor of the same multiplicity.*

Suppose that  $X$  has block diagonal form, and  $A$  has corresponding block matrix form:

$$(5) \quad X = \begin{pmatrix} X_1 & 0 & \cdots \\ 0 & X_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \\ \vdots & & \ddots \end{pmatrix}.$$

Then, by (2),

$$(6) \quad X_i A_{ij} X_j' = A_{ij} = -A_{ji}' \text{ for all } i, j.$$

Hence  $A_{ij} X_j' = X_i^{-1} A_{ij}$  and so, more generally,

$$(7) \quad A_{ij} f(X_j)' = f(X_i^{-1}) A_{ij}$$

for any polynomial  $f(t)$ .

**Notation 2.5.** Let  $c_Y(t)$  denote the characteristic polynomial of the square matrix  $Y$ .

**Lemma 2.6.** *If*

$$(8) \quad (c_{X_i^{-1}}(t), c_{X_j}(t)) = 1,$$

*then  $A_{ij} = 0$ .*

*Proof.* Taking  $f(t) = c_{X_i^{-1}}(t)$  in (7), we get  $A_{ij} c_{X_i^{-1}}(X_j)' = 0$ . However, in view of (8),  $c_{X_i^{-1}}(X_j)$  is nonsingular, whence  $A_{ij} = 0$ .  $\square$

Elementary divisors  $f_1(t)$ ,  $f_2(t)$  of  $X$  are powers of irreducible monic polynomials  $g_1(t)$ ,  $g_2(t)$ . We say that  $f_1(t)$  and  $f_2(t)$  are *related* if  $g_2(t) = g_1(t)$  or  $g_1^-(t)$ .

By the theory of elementary divisors, we may choose the blocks  $X_i$  in (5) in such a way that elementary divisors  $f_1(t)$ ,  $f_2(t)$  of  $X$  are elementary divisors of the same  $X_i$  if, and

only if, they are related. With such a choice of the  $X_i$ , Lemma 2.6 shows that  $A$  has corresponding block diagonal form

$$\begin{pmatrix} A_{11} & 0 & \cdots \\ 0 & A_{22} & \\ \vdots & & \ddots \end{pmatrix}.$$

In this way the original problem for  $X$  is reduced to the same problem for the individual blocks  $X_i$ . We may therefore assume:

**Assumption 2.7.** There exists a monic irreducible polynomial  $g(t) \neq t$  such that every elementary divisor of  $X$  is a power of  $g(t)$  or  $g^-(t)$ .

**Case 1:**  $\mathbf{g}(\mathbf{t}) \neq \mathbf{g}^-(\mathbf{t})$ . We may assume in (5) that

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where  $c_{X_1}(t)$ ,  $c_{X_2}(t)$  are powers of  $g(t)$ ,  $g^-(t)$  respectively. By Lemma 2.1,  $X_2$  is similar to  $X_1^{-1}$ . We may therefore assume further that

$$(9) \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & (X_1^{-1})' \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}' & A_{22} \end{pmatrix}.$$

By Lemma 2.6,  $A_{11} = A_{22} = 0$ . Also, by (6),  $X_1 A_{12} X_1^{-1} = A_{12}$ , i.e.  $X_1$  commutes with  $A_{12}$ .

Now let

$$(10) \quad Y = \begin{pmatrix} A_{12} & 0 \\ 0 & I_m \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where  $I_m$  is the  $m \times m$  unit matrix, and  $m = \deg c_{X_1}(t)$ . Then

$$Y^{-1}XY = X, \quad YJY' = A,$$

showing that, with  $X$  as in (9), there is just one orbit under the action (3), represented by the matrix  $J$  in (10). Expressed differently, the conjugacy class of  $X$  in  $\mathrm{GL}_{2m}(F)$  intersects each symplectic subgroup in a single conjugacy class of the latter.

**Case 2:**  $\mathbf{g}(\mathbf{t}) = \mathbf{g}^-(\mathbf{t})$ .

In general, the elementary divisors of  $X$  may be arbitrary powers of  $g(t)$  with arbitrary multiplicities. We now impose the condition that  $X$  be semisimple:

**Assumption 2.8.**  $X \in \mathrm{GL}_{2m}(F)$  has the single irreducible elementary divisor  $g(t) \neq t$  with multiplicity  $n$ .

We may therefore assume that

$$(11) \quad X = \mathrm{diag}(\underbrace{R, \dots, R}_n),$$

where

$$(12) \quad c_R(t) = g(t).$$

Since  $c_R(t)$  is irreducible, the matrices

$$f(R) \quad (f(t) \in F[t])$$

form a field

$$K \cong F[t]/g(t)F[t]$$

and every matrix that commutes with  $R$  is in  $K$ . It follows that the centralizer of  $X$  in  $\mathrm{GL}_{2m}(F)$  consists of the nonsingular  $n \times n$  block matrices

$$(13) \quad B = (f_{ij}(R))_{i,j=1,\dots,n},$$

for polynomials  $f_{ij}(t) \in F(t)$ . These matrices form a group that we may identify with  $\mathrm{GL}_n(K)$ .

If  $\deg g(t) = 1$ , then  $g(t) = t \pm 1$  (since  $g(t) = g^{-1}(t)$ ), the matrix  $R$  is a  $1 \times 1$  matrix ( $\pm 1$ ) and  $X = \pm I_{2m}$ . The centralizer of  $X$  is  $\mathrm{GL}_{2m}(F)$  and the nonsingular  $2m \times 2m$  skew-symmetric matrices form a single orbit under its action. We assume from now on that  $\deg g(t) \geq 2$ .

Let

$$(14) \quad A = (A_{ij})_{i,j=1,\dots,n}$$

be the block form of  $A$  corresponding to (11). The equation  $A = XAX'$  in (2) is then equivalent to the set of equations

$$(15) \quad RA_{ij}R' = A_{ij}.$$

Now, since  $g(t) = g^{-1}(t)$ ,  $R$  is similar to  $R^{-1}$  and so

$$(16) \quad R' = T^{-1}R^{-1}T$$

for some  $T \in \mathrm{GL}_{2m/n}(F)$ . Thus, we may rewrite (15) as

$$R(A_{ij}T^{-1}) = (A_{ij}T^{-1})R,$$

whence  $A$  has the form

$$(17) \quad A = (f_{ij}(R)T).$$

We write this equation as

$$(18) \quad A = B\mathcal{T} \quad \text{where } B = (f_{ij}(R)) \text{ and } \mathcal{T} = \mathrm{diag}(\underbrace{T, \dots, T}_n).$$

Now, the mapping  $K \rightarrow K$  defined by

$$\phi(R) \mapsto \phi(R^{-1}) \quad (\phi(t) \in F[t])$$

is a field automorphism of  $K$  of order 2, since  $R \neq R^{-1}$ . For a matrix

$$Y = (\phi_{ij}(R)) \in \mathrm{GL}_n(K),$$

we define

$$Y^* = (\phi_{ij}(R^{-1}))^{\mathrm{tr}},$$

where  $\text{tr}$  denotes transpose qua  $n \times n$  matrix over  $K$  and *not* qua  $2m \times 2m$  matrix over  $F$ , i.e.  $Y^* = (\phi_{ji}(R^{-1}))$ . Accordingly,  $Y$  is called *Hermitian* if  $Y^* = Y$  and  $Y_1, Y_2$  are said to be *\*-congruent* if  $Y_2 = CY_1C^*$  for some  $C \in \text{GL}_n(K)$ .

The following two results are proved by routine calculations:

**Lemma 2.9.** (i) If  $A = B\mathcal{T}$  as in (18) and  $Y \in \text{GL}_n(K)$ , then  $YAY' = YBY^*\mathcal{T}$ .  
(ii) If  $A = B\mathcal{T}$  as in (18) and  $T$  is skew-symmetric, then  $A$  is skew-symmetric if, and only if,  $B$  is Hermitian.

*Proof.* (i) We may write  $Y$  as a block matrix  $(\phi_{ij}(R))$  for some  $\phi_{ij}(t) \in F[t]$ . Then

$$\begin{aligned} YAY' &= (\phi_{ij}(R))(f_{ij}(R)T)(\phi_{ij}(R))' \\ &= (\sum_{\lambda,\mu} \phi_{i\lambda}(R)f_{\lambda\mu}(R)T\phi_{j\mu}(R')) \\ &= (\sum_{\lambda,\mu} \phi_{i\lambda}(R)f_{\lambda\mu}(R)\phi_{j\mu}(R^{-1})T), \\ &= YBY^*\mathcal{T} \end{aligned}$$

by (16).

(ii) By (18),  $A' = \mathcal{T}'B'$ , and since  $T$  is skew-symmetric we have

$$\begin{aligned} -A' &= (-T'f_{ji}(R')) = (Tf_{ji}(R')) && (\text{as } T' = -T) \\ &= (f_{ji}(R^{-1})T) && (\text{by (16)}). \end{aligned}$$

So  $A = -A'$  if, and only if,  $f_{ij}(R)T = f_{ji}(R^{-1})T$  for all  $i, j$ . Since  $T$  is invertible, this holds if, and only if,

$$B = (f_{ij}(R)) = (f_{ji}(R^{-1})) = B^*.$$

□

Thus, *provided that  $T$  can be chosen skew-symmetric*, our conjugacy class problem in the present case reduces to a classification problem for Hermitian forms over the extension  $K$  of  $F$ . Since  $T$  can obviously be replaced in (17) by  $h(R)T$ , where  $h(R)$  is any nonzero (and hence nonsingular) element of  $K$ , the following result shows that such a choice of  $T$  is always possible.

**Lemma 2.10.** If  $g(t) \neq t \pm 1$ , then there exists a nonzero element  $h(R)$  of  $K$  such that  $h(R)T$  is skew-symmetric.

*Proof.* (16) can be written  $RTT' = T$ . Transposing, we get  $RT'R' = T'$ , whence

$$(19) \quad R' = (T')^{-1}R^{-1}(T').$$

Comparing with (16), we deduce that  $T'T^{-1}$  commutes with  $R$ , whence

$$T' = f(R)T$$

for some  $f(R) \in K$ .

Now, if both  $RT$  and  $T$  were symmetric, we would have

$$RT = (RT)' = T'R' = TR' = R^{-1}T$$

by (16). But this implies that  $R = R^{-1}$  and so  $R^2 = I$ , contrary to the assumption that  $g(t) \neq t \pm 1$ .

It follows that at least one of  $T$  and  $RT$  — let us say  $T$  itself — is *not* symmetric. But then

$$T - T' = (1 - f(R))T$$

is nonzero and skew-symmetric, as required.  $\square$

### 3. SUMMARY

We wish to determine a complete, irredundant set of conjugacy class representatives for the semisimple elements of the symplectic groups  $\mathrm{Sp}_{2m}(F)$ , where  $F$  is an arbitrary field of characteristic not 2. To do so, it suffices to consider the elements whose characteristic polynomial is divisible only by powers of  $g(t)$  and  $g^-(t)$ , for some irreducible  $g(t) \in F[t]$ . We first choose a set  $R_1, R_2, \dots, R_k$  of representatives of conjugacy classes of such elements in  $\mathrm{GL}_{2m}(F)$ , and discard any such  $R_i$  that do not preserve a symplectic form.

In Case 1 of Section 2,  $g(t) \neq g^-(t)$ . Then the  $\mathrm{GL}_{2m}(F)$ -conjugacy class of  $R_i$  meets  $\mathrm{Sp}_{2m}(F)$  in a unique conjugacy class.

In Case 2,  $g(t) = g^-(t)$ . The only  $R_i$  for which  $g(t)$  has degree 1 are  $\pm I_{2m}$ . Let  $R_i = X$  be as in (11), where  $R$  has characteristic polynomial  $g(t)$  of degree greater than 1, and let  $K$  denote the field isomorphic to the set of all polynomials in  $R$ . Then each congruence class of Hermitian forms on  $K^{2m/n}$  corresponds to an  $\mathrm{Sp}_{2m}(F)$ -conjugacy class of matrices that are similar to  $R_i$ . In particular, if  $F$  is finite then there is only one such class, and so once again the  $\mathrm{GL}_{2m}(F)$ -conjugacy class of  $R_i$  meets  $\mathrm{Sp}_{2m}(F)$  in a unique conjugacy class.

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